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ELLIPTIC ADAPTIVE GRID GENERATION AND AREA EQUIDISTRIBUTION

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SUMMARY

The derivation of elliptic adaptive grid control functions which satisfy the area equidistribution concept is presented. The resulting expressions are derived without approximation and are shown to provide explicit control over cell area distributions. A modification to the equidistribution concept which yields control functions that enable additional control of the near-boundary grid resolution is also proposed. A computer code which incorporates these control functions has been developed and is applied to a series of complex fluid flows to demonstrate the validity and utility of the derived expressions. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: grid generation; elliptic; solution adaptive; equidistribution

1. INTRODUCTION

The accuracy of simulations employing computational fluid dynamics (CFD) is highly dependent upon the quality of the grid used in the numerical analysis. Large flow field gradients in regions of large grid spacing lead to large discretization errors and poor solutions. In the analysis of complex fluid flows, it is not always known *a priori* where significant flow field gradients will occur, hence, the use of adaptive grids becomes especially important. Such techniques make use of a solution error estimate to redistribute grid points so as to globally minimize this error, ensuring that the more significant gradients, wherever they may occur, are better resolved.

Adaptive grid schemes for structured grids are typically classified as belonging to one of three groups which consist of variational methods, partial differential equation methods and algebraic methods. The variational method, as introduced by Brackbill and Saltzman [1], involves the minimization of a function consisting of grid smoothness, orthogonality and adaption measures via the solution to a system of elliptic partial differential equations. The sound mathematical basis of the method is attractive, however, its complexity renders it more costly and difficult to use, as selection of optimal values for the relative weightings can become a time consuming process. Kreis *et al*. [2] have explored this scaling problem and indicate that poorly chosen weights can lead to a mathematically ill-posed problem and, hence, no solution at all.

A related, though more tractable approach, involving the solution to Poisson's equation was first presented for two-dimensions by Winslow [3] and later by Anderson and Steinbrenner [4]

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and Anderson [5], and subsequently extended to three-dimensions by Kim and Thompson [6]. Within the well-known elliptic grid generation equations, modification of the non-homogeneous terms, otherwise known as the grid control functions, converts the scheme into an effective adaptive grid generation tool. With proper formulation of the control functions, it is possible to direct the placement of grid points to meet the particular needs of the problem at hand. Therein lies both the difficulty and power of the method, as there is complete flexibility in this regard.

The most common adaptive strategy falls into the category of algebraic schemes which have as their basis the concept of equidistribution, or some variation thereof, as introduced by Dwyer *et al*. [7]. Ease of implementation, computational speed and robustness are characteristics of the algebraic approach, however, grid smoothness, skewness and orthogonality can become significant problems as this approach typically involves uncoupled sweeps of a one-dimensional adaption operator. Numerous investigators [8–12] have attempted to address these shortcomings in different ways with varying degrees of success. From these efforts, it is apparent that improved grid quality was achieved by use of a bi-directional adaption operator, which employs an area-based rather than arc length-based concept, its simpler one-dimensional counterpart. Furthermore, only a single adaption function need be specified for multi-dimensional problems.

Partial differential equation methods remain attractive due to the fact that they represent a good compromise between complexity and performance. They are less susceptible to the problems exhibited by algebraic schemes in that the very nature of the governing equations themselves tends to yield smooth grids. Winslow was the first to recognize the possibility of using these equations to generate adaptive grids, his particular approach involving the introduction of a 'diffusion coefficient' into Laplace's equation. This 'diffusion coefficient' is then related to solution gradients to affect the cell area distribution. In a different vein, Anderson and Steinbrenner based their development of grid control functions on the concept of arc equidistribution. With the elliptic equations recast as what they refer to as non-linear equidistribution equations, they show that, under rather restrictive assumptions of local grid orthogonality and vanishing curvature, the equations simplify considerably to the point where they become identical to the arc equidistribution equations. This permits them to infer the form of the control functions which are each shown to consist of a single term. Without these assumptions, however, no control functions can be provided.

In a later paper, Anderson makes a strong case for the need of an area-based rather than arc length-based formulation of the control functions to ameliorate the tendency of the latter to produce skewed grids. Indeed, investigations by the current author into algebraic methods employing one-dimensional and multi-dimensional adaption operators supports this assertion. In this light, Anderson re-examines Winslow's 'diffusion' approach and formally presents control functions of the standard form which result from the method. These control functions, which each include two terms, are shown to provide for improved grid quality over the arc length based scheme. However, these new control functions are difficult to characterize in that they satisfy no explicit constraint; they provide a means to influence rather than control the area distribution. The question then arises as to whether control functions which permit direct control of the distribution can be derived.

In the current work, a mathematically rigorous investigation into the derivation of a set of elliptic adaptive grid control functions which satisfy the area equidistribution concept is undertaken; exact expressions for the control functions are derived without approximation as no assumptions about the nature of the grid are made. Furthermore, the formulation is extended to permit additional control of the near-boundary grid. To demonstrate the validity of the results, a computer code has been developed which incorporates the derived control functions within an elliptic grid generation scheme. This code is applied to a supersonic inlet followed by application to a gas jet nose tip.

2. CONTROL FUNCTION DERIVATION

In two-dimensional transformed space (ξ, η), the elliptic grid generation equations [13] are

$$£\vec{r} + g(P\vec{r}_\xi + Q\vec{r}_\eta) = 0, \tag{1}$$

where $\vec{r} = (x, y)$ and g represents the square of the Jacobian of the transformation from (x, y) to (ξ, η) space. P and Q are the control functions which provide the avenue through which the placement of the grid points may be controlled. In tensor form, the operator $£$ is defined as

$$£ = gg^{ij}\frac{\partial^2}{\partial\xi^i\,\partial\xi^j}, \tag{2}$$

where

$$g^{11} = g_{22}/g, \qquad g^{12} = -g_{12}/g, \qquad g^{22} = g_{11}/g, \qquad g = g_{11}g_{22} - g_{12}^2, \tag{3}$$

and

$$g_{ij} = \frac{\partial\vec{r}}{\partial\xi^i}\cdot\frac{\partial\vec{r}}{\partial\xi^j}. \tag{4}$$

The solution to this equation has been highly developed for grid generation of surfaces in two dimensions. The goal of this work is to extend this capability by determining a set of control functions which will turn this into an adaptive grid generation scheme which enforces the area equidistribution concept. Hence, a relationship between the control functions and the adaption function must be established. The first step in deriving such a relationship is to solve Equation (1) for P and Q. To accomplish this, the scalar product of vector Equation (1) with $\vec{r}_\xi$ and $\vec{r}_\eta$ is evaluated, which results in two simultaneous equations for P and Q, the solution for which is

$$P = \frac{1}{g^2}(g_{12}\vec{r}_\eta\cdot£\vec{r} - g_{22}\vec{r}_\xi\cdot£\vec{r}), \tag{5}$$

and

$$Q = \frac{1}{g^2}(g_{12}\vec{r}_\xi\cdot£\vec{r} - g_{11}\vec{r}_\eta\cdot£\vec{r}). \tag{6}$$

To avoid rather involved algebraic manipulation of these equations, it is useful to recast them in terms of Christoffel symbols using the following equation [14]

$$\frac{\partial^2\vec{r}}{\partial\xi^i\,\partial\xi^j} = \Gamma^m_{ij}\frac{\partial\vec{r}}{\partial\xi^m}. \tag{7}$$

This permits the scalar products to be recast within the parenthesis using

$$\frac{\partial\vec{r}}{\partial\xi^n}\cdot£\vec{r} = gg_{nm}g^{ij}\Gamma^m_{ij}, \tag{8}$$

which, after some algebraic manipulation, yields

$$P = -\frac{1}{g}(g_{22}\Gamma^1_{11} - 2g_{12}\Gamma^1_{12} + g_{11}\Gamma^1_{22}), \tag{9}$$

$$Q = -\frac{1}{g}(g_{22}\Gamma^2_{11} - 2g_{12}\Gamma^2_{12} + g_{11}\Gamma^2_{22}). \tag{10}$$

The concept of area equidistribution is now introduced to serve as a constraint such that, when the resulting equations for P and Q are substituted back into Equation (1), an adaptive set of elliptic equations will result.

The area equidistribution concept is stated mathematically as

$$f\sqrt{g} = \text{constant}, \tag{11}$$

where the adaption function, f, is based on solution flow field gradients, which provides the error indicator on which the adaption is to be based. Application of this concept will dictate smaller cell areas in regions of large gradients and vice versa. Differentiation of this equation with respect to ξ and η yields

$$-f_\xi/f = g_\xi/2g, \tag{12}$$

$$-f_\eta/f = g_\eta/2g. \tag{13}$$

The derivative of g may be expressed in terms of the Christoffel symbols using the following equation [14]

$$\frac{1}{2g}\frac{\partial g}{\partial \xi^j}\Gamma^i_{ij}, \tag{14}$$

which enables the previous equations to be recast as

$$-f_\xi/f = \Gamma^1_{11} + \Gamma^2_{12}, \tag{15}$$

$$-f_\eta/f = \Gamma^1_{12} + \Gamma^2_{22}, \tag{16}$$

respectively. To introduce these equations into the expressions for P and Q, Equation (16) multiplied by g_{12} is subtracted from Equation (15) multiplied by g_{22} to yield

$$g_{22}\Gamma^1_{11} - 2g_{12}\Gamma^1_{12} = -g_{22}f_\xi/f + g_{12}f_\eta/f - g_{22}\Gamma^2_{12} - g_{12}(\Gamma^1_{12} - \Gamma^2_{22}). \tag{17}$$

In a similar manner, Equation (16) multiplied by g_{11} is subtracted from Equation (15) multiplied by g_{12} to yield

$$g_{11}\Gamma^2_{22} - 2g_{12}\Gamma^2_{12} = g_{12}f_\xi/f - g_{11}f_\eta/f - g_{11}\Gamma^1_{12} - g_{12}(\Gamma^2_{12} - \Gamma^1_{11}). \tag{18}$$

Note that these equations have been arranged such that the terms of the left-hand-side are common to the terms in the expressions for the control functions given in Equations (9) and (10), respectively. If these equations are substituted into the equations for P and Q, we obtain

$$P = \frac{1}{g}(g_{22}(f_\xi/f) - g_{12}(f_\eta/f) + g_{22}\Gamma^2_{12} + g_{12}(\Gamma^1_{12} - \Gamma^2_{22}) - g_{11}\Gamma^1_{22}), \tag{19}$$

$$Q = \frac{1}{g}(g_{11}(f_\eta/f) - g_{12}(f_\xi/f) + g_{22}\Gamma^2_{11} + g_{12}(\Gamma^2_{12} - \Gamma^1_{11}) + g_{11}\Gamma^1_{12}). \tag{20}$$

Substituting for the Christoffel symbols using Equation (7) and applying Equation (4), the control functions can be rewritten as

$$P = \frac{1}{g}\,(g_{22}(f_\xi/f) - g_{12}(f_\eta/f) - \vec{r}_\xi\cdot\vec{r}_{\eta\eta} + \vec{r}_\eta\cdot\vec{r}_{\xi\eta}), \tag{21}$$

and

$$Q = \frac{1}{g}\,(g_{11}(f_\eta/f) - g_{12}(f_\xi/f) - \vec{r}_\eta\cdot\vec{r}_{\xi\xi} + \vec{r}_\xi\cdot\vec{r}_{\xi\eta}), \tag{22}$$

which are the exact form of the adaptive control functions derived without approximation.

For the arc length-based approach, previous researchers [4,6] have used control functions of the form

$$P = \frac{1}{g}\,g_{22}(f_\xi/f), \tag{23}$$

and

$$Q = \frac{1}{g}\,g_{11}(f_\eta/f), \tag{24}$$

which contain only the leading terms of Equations (21) and (22), respectively. The expressions derived by Anderson from Winslow's 'diffusion' approach have the form

$$P = \frac{1}{g}\,(g_{22}(f_\xi/f) - g_{12}(f_\eta/f)), \tag{25}$$

and

$$Q = \frac{1}{g}\,(g_{11}(f_\eta/f) - g_{12}(f_\xi/f)). \tag{26}$$

Comparison of these equations to (21) and (22) demonstrates conclusively that Winslow's approach represents only an approximate constraint on the cell area distribution. For the case of a uniform adaption function, Equations (25) and (26) are identically zero, which will yield a Laplacian grid. In contrast, the control functions offered in Equations (21) and (22) do not necessarily tend to zero. The nature of the grid which results from application of these control functions is investigated herein. First, Equation (14) is expanded for $j = 1$, which yields

$$\frac{1}{2g}\frac{\partial g}{\partial \xi} = \Gamma^1_{11} + \Gamma^2_{22}. \tag{27}$$

Next the scalar product of $\vec{r}_\xi$ is formed with Equation (1) to obtain

$$\vec{r}_\xi\cdot \pounds r + gg_{11}P + gg_{12}Q = 0. \tag{28}$$

Substituting for the scalar product using Equation (8) and the control functions using Equations (19) and (20) yields, after some algebraic manipulation,

$$\Gamma^1_{11} + \Gamma^2_{12} = 0. \tag{29}$$

A similar result also exists for the η differentiation of Equation (14). This demonstrates that the control functions derived in Equations (19) and (20) satisfy the area equidistribution concept, in that they enforce a constant cell area for a uniform adaption function. Whether or not this is the preferred behavior is subject to debate, however, such a formulation presents an avenue through which the cell area distribution may be 'fine-tuned' to meet specific problem requirements.

3. RESOLUTION ENHANCEMENT

Adaptive grid generation has been shown to significantly improve the resolution of important flow field structures in highly non-linear flows. Typically, as the definition of the adaption function and related parameters are adjusted to optimize the resolution of shock waves, a coarsening of the grid prescribed to resolve boundary layers near solid walls usually results. This especially troublesome tendency has not been adequately addressed within the literature and in this author's opinion has contributed to the general lack of applicability of the technology.

To enhance the applicability of elliptically generated adaptive grids, a modified area equidistribution concept, which affords the user the ability to control near-boundary resolution, is proposed. The modified concept has the form

$$f\sqrt{g} = h \cdot \text{constant}, \tag{30}$$

where $h(\xi, \eta)$ is a user defined function. The manner in which h must be defined to achieve increased near-boundary resolution is easily discerned. First, in order to recover the standard area equidistribution concept away from the wall, h is prescribed to take on a value of unity at an arbitrary distance away from the boundary. Defining the boundary value, h_o, to be less than unity provides for a smaller cell area in accordance with Equation (30). In practice, this function is computed within the domain using a modified form of transfinite interpolation.

The derivation of the control functions based on this concept proceeds as outlined previously, where the modified expression is differentiated and cast in the form given in Equation (12) such that

$$-f_\xi/f \to -f_\xi/f + h_\xi/h = g_\xi/2g, \tag{31}$$

and the remainder of the derivation remains the same. The control functions which incorporate the modified form of the area equidistribution concept are obtained by substituting this expression and its counterpart from the η differentiation for the terms in Equations (21) and (22). Obviously with $h = 1$ the original equations are recovered. However, with a suitably defined distribution function, the near-wall resolution can be controlled and adjusted to meet user requirements.

4. THE ADAPTIVE GRID SCHEME

The elliptic adaptive grid methodology described represents only a single element within a complete adaptive algorithm as the adaption of the interior points and is governed by the solution to Equation (1). Additional constraints which exist for boundary points preclude direct use of this technique and other methods must be devised. This also represents a significant problem in that orthogonality and smoothness are especially important issues in the near-boundary region. Methods included within the computer code to provide for boundary point movement can be found in Reference [12].

5. ADAPTION FUNCTION

In the applications to follow, the adaption function is based on a function W, which is computed from a linear weighting of N user selected flow field variables according to the equation

$$W = \frac{\sum_{n=1}^{N} \phi_n \bar{w}_n}{\sum_{n=1}^{N} \phi_n}, \tag{32}$$

where the dimensionless flow field variable $\bar{w}_n$ has the weighting of ϕ_n. The adaption function is then computed as

$$f = W_o \phi_f + |\nabla W|^2, \tag{33}$$

where W_o is the area weighted average of $|\Delta W|^2$ and ϕ_f is a user defined weighting constant which controls the strength of the adaption.

6. APPLICATIONS

6.1. Supersonic inlet

This application involves the calculation of an inviscid internal flow through a two-dimensional duct with a 11.3° compression ramp. Freestream conditions used for this case were $M_\infty = 1.95$, $T_\infty = 530°$ R and $p_\infty = 2116$ psf. The initial and final adapted grids are shown in Figure 1. A slight clustering was prescribed on the upper and lower boundaries of the initial grid. The adaption function was based on the Mach number only with $\phi_f = 0.2$ and $h_o = 1$.

Extensive point movement is evident due to the system of shocks and the corner expansion fan has also become more resolved. The initial clustering near the upper and lower walls has vanished due to the lack of significant gradients. A close-up view of the adapted grid is shown in Figure 2.

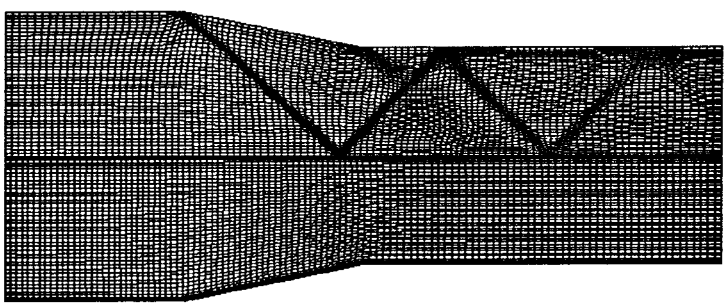

Figure 1. Adapted and initial grid for supersonic inlet.

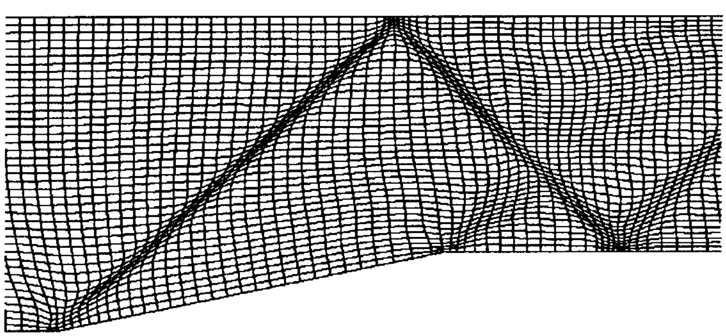

Figure 2. Adapted grid for supersonic inlet, expanded view.

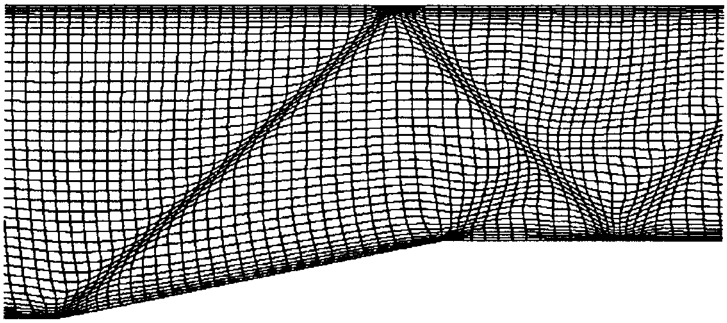

Figure 3. Adapted grid with clustering enhancement, $h_o = 0.5$.

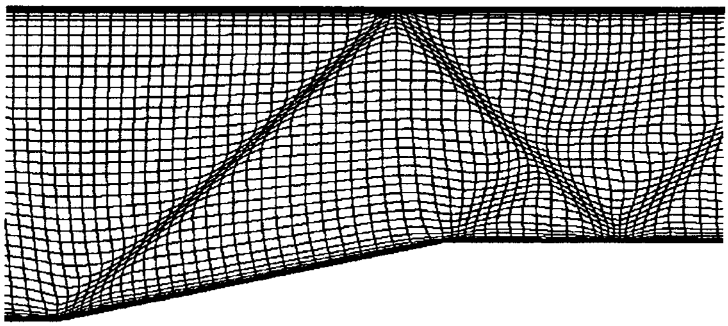

Figure 4. Adapted grid with clustering enhancement, $h_o = 0.25$.

To demonstrate the resolution enhancement offered by the modified form of the area equidistribution concept, the adaption was repeated with grids generated corresponding to $h_o = 0.5$ and 0.25. These grids are shown in Figures 3 and 4, respectively.

Note that although the increased resolution is artificially induced in the transverse direction, the effects of the adaption function gradients continue to influence the streamwise point distributions in these areas.

6.2. Gas jet nose tip

A more rigorous test of the method is achieved with the gas jet nose tip problem. In the past, the concept of counterflow mass injection has been investigated as a means to alleviate pressure and thermal loads over blunt shapes in supersonic flows. Experimental and numerical investigations have indicated the flow field to be quite complex with multiple shocks and large regions of subsonic and separated flow present. Due to the complex nature of this flow, the analysis is ideally suited to the use of adaptive grid generation.

The configuration analyzed is a blunted, 2-inch diameter hemisphere cylinder having a 0.264-inch diameter jet orifice at the nose. The freestream conditions used were $M_\infty = 2.5$, $T_\infty = 231.1°$ and $\mathscr{R}_e = 1.4 \times 10^6$ based on body diameter. Turbulent flow was assumed. The analysis was performed at a jet pressure ratio (jet total to freestream pitot pressure) of 2.08. The jet supply was taken from the tunnel supply and issued at sonic velocity. The configuration analyzed in this application is identical to that tested by Finley [15]. The final adapted grid for this analysis was obtained using an adaption function composed of the Mach number and vorticity magnitude with $\phi_f = 0.20$ and $h_o = 1.0$ and is shown in Figure 5.

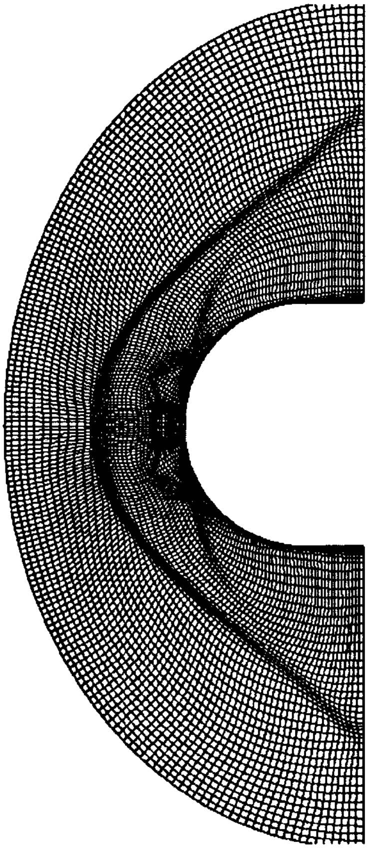

Figure 5. Adapted grid for gas jet nose tip analysis.

Readily apparent is the realignment of grid points due to the bow shock, Mach disk and barrel shock as well as within the shear layer adjacent to the recirculation region. Note that the weak recompression shock at reattachment is also discernible. A close-up view of the jet region is shown in Figure 6.

As in the previous case, the adaption has been repeated to demonstrate the enhanced clustering which may be obtained using the control functions derived from the modified formulation. The resulting grid, in which additional near-wall clustering has been enhanced at the body surface only, is shown in Figure 7.

7. CONCLUSIONS

This effort presents the results of a rigorous mathematical investigation into the derivation of elliptic adaptive grid control functions which satisfy the area equidistribution concept. The derived control functions differ from those offered by previous researchers, in that they provide for direct control of the resulting cell area distributions. Furthermore, the formulation provides a convenient means to permit additional user control over the near-boundary grid resolution. Results presented with the derived control functions were shown to validate the utility of the expressions in producing high quality adapted grids.

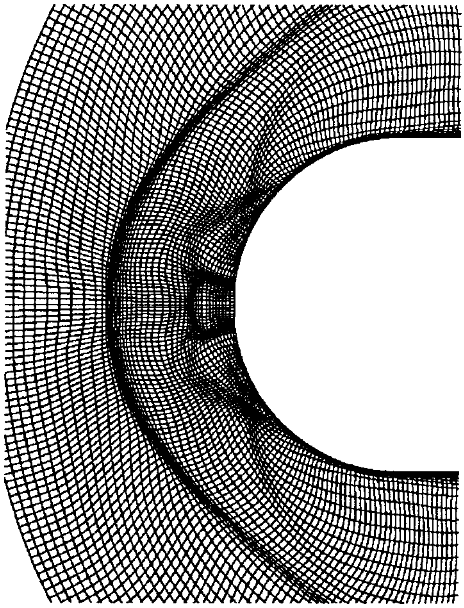

Figure 6. Adapted grid for gas jet nose tip, jet region.

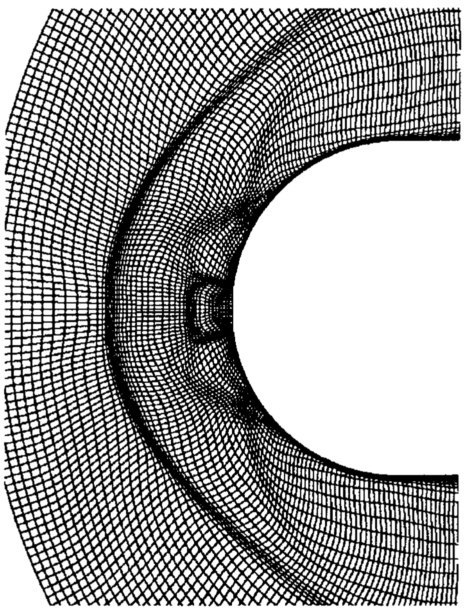

Figure 7. Adapted grid with clustering enhancement, $h_o = 0.5$.

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